1) Solution:

(i) General heat transfer rate is given by

$$Q = -KA\frac{dt}{dx}$$

$$\Rightarrow \quad Q\,dx = -KA\,dt$$

$$\Rightarrow \quad Q\,dx = -(K_0 t^2)(A)\,dt$$

Integrating both sides with limits of $x: 0 \to L$, $t: t_1 \to t_2$

$$\Rightarrow \quad \int_0^L Q\,dx = -\int_{t_1}^{t_2} (K_0 t^2)(A)\,dt$$

$$\Rightarrow \quad QL = -K_0 A \frac{(t_2^3 - t_1^3)}{3}$$

$$\Rightarrow \quad Q = \frac{K_0 A (t_1 - t_2)(t_1^2 + t_2^2 + t_1 t_2)}{3L}$$

Expression to find steady state conduction (1)

(ii) Let $Q = \frac{K_m A (t_1 - t_2)}{L}$ -- ②

From equations ①, ②, we get

$$K_m = K_0\left(\frac{t_1^2 + t_2^2 + t_1 t_2}{3}\right) \quad \text{-- ③}$$

But general variation in conductivity is given by $K = K_0 t^2$ -- ④

① comparing eq's ③, ④ we get mean temperature at which mean thermal conductivity is evaluated as

$$t_m = \sqrt{\frac{t_i^2 + t_2^2 + t_i t_2}{3}}$$

2. SOLUTION

Conduction of heat in given fire tube is in transient state since temperature at any point is varying with time because of difference in heat flow at inner and outer surfaces.

(i) Heat flow rate is given by

$$Q = -KA\left(\frac{\partial t}{\partial x}\right).$$

Given $t = 800 + 1000r - 5000r^2$

$$\Rightarrow Q = -K(2\pi r)(l)\left(\frac{\partial}{\partial r}(800 + 1000r - 5000r^2)\right)$$

$$= -K(2\pi r)(l)(1000 - 10000r)$$

$$\Rightarrow Q/l = -K(2\pi r)(1000 - 10000r)$$

Rate of heat flow per unit length at inside surface @

$$Q/l\Big|_{r=0.3m} = -(58)(2)\left(\frac{22}{7}\right)(0.3)(1000 - 10000(0.3))$$

$$\Rightarrow \left.\frac{Q}{l}\right|_{r=0.3m} = 218,654.84 \text{ W/m} = 218.65 \text{ kW/m}$$

Rate of heat flow per unit length at outer surface

$$\left.\frac{Q}{l}\right|_{r=0.5} = -(58)(2)\left(\frac{22}{7}\right)(0.5)(1000-10000(0.5))$$

$$= 728,849.49 \text{ W/m} = 728.85 \text{ kW/m}$$

(ii) Rate of heat storage per unit length

= Rate of heat inflow (at inner surface) - Rate of heat outflow (at outer surface)

$$= 218.65 - 728.85 = \underline{-510.2 \text{ kW/m}}$$

negative sign indicates that heat energy is lost by the firetube.

(iii) Rate of temperature change for a cyllindrical object is given by

$$\frac{dt}{d\tau} = \alpha\left[\frac{d^2t}{dr^2} + \frac{1}{r}\frac{dt}{dr}\right], \quad \alpha = 0.004 \text{ m}^2/\text{h}$$

At inner surface:

$$\left.\frac{d^2t}{dr^2}\right|_{r=0.3} = \left.\frac{d^2t}{dr^2}\right|_{r=0.5} = -10000$$

$$\left.\frac{dt}{dr}\right|_{r=0.3} = 1000 - 10000(0.3)$$

$$= \cancel{4000} -2000$$

$$\Rightarrow \left.\frac{dt}{d\tau}\right|_{r=0.3} = 0.004\left[-10000 + \frac{1}{0.3}(-2000)\right]$$

$$= -66.67\,^\circ C/h$$

[decreasing]

At outer surface:

$$\left.\frac{d^2t}{dr^2}\right|_{r=0.5} = -10{,}000$$

$$\left.\frac{dt}{dr}\right|_{r=0.5} = 1000 - 10000(0.5)$$

$$= -4000$$

$$\Rightarrow \left.\frac{dt}{d\tau}\right|_{r=0.5} = 0.004\left[-10000 + \frac{1}{0.3}(-4000)\right]$$

$$= -93.33\,^\circ C/h.$$

[decreasing]

③ SOLUTION:

General one-dimensional conduction equation is given by

$$Q = -KA\frac{dt}{dx}$$

Here area varies with distance as $A = \frac{\pi}{4}D^2 = \frac{\pi}{4}(cx)^2 = \frac{\pi}{4}c^2x^2$

So, we have

$$Q = -K\left(\frac{\pi}{4}c^2x^2\right)\frac{dt}{dx}$$

$$\Rightarrow \quad -K\,dt = \frac{4Q\,dx}{\pi c^2 x^2}$$

Integrating both sides with limits of $t : t_1 \rightarrow t$ [t = temperature at distance 'x']

$x : x_1 \rightarrow x$

$$\Rightarrow \quad \int_{t_1}^{t} -K\,dt = \int_{x_1}^{x} \frac{4Q\,dx}{\pi c^2 x^2}$$

$$\Rightarrow \quad -K\int_{t_1}^{t} dt = \frac{4Q}{\pi c^2}\int_{x_1}^{x}\frac{dx}{x^2}$$

$$\Rightarrow \quad -K(t-t_1) = \frac{4Q}{\pi c^2}\left[\frac{-1}{x} - \left(\frac{-1}{x_1}\right)\right]$$

$$\Rightarrow \quad -K(t-t_1) = \frac{4Q}{\pi c^2}\left[\frac{1}{x_1} - \frac{1}{x}\right]$$

On solving for 't', we get

$$\underline{t} = t_1 - \frac{4Q}{\pi c^2 K}\left[\frac{1}{x_1} - \frac{1}{x}\right]$$

Here 't' is temperature at distance 'x'

④ SOLUTION

The expression for conduction of heat through a spherical shell is given by

$$Q = \frac{4\pi K r_1 r_2 (t_1 - t_2)}{(r_2 - r_1)}$$

$$Q = -KA\frac{dt}{dr}$$

where, $A = 4\pi r^2$, $K = K_0(1+\alpha t+\beta t^2)$

$$\Rightarrow Q = -K_0(1+\alpha t+\beta t^2)(4\pi r^2)\left(\frac{dt}{dr}\right)$$

$$\Rightarrow \frac{Q\,dr}{r^2} = (-K_0)(4\pi)(1+\alpha t+\beta t^2)\,dt$$

Integrating both sides: [limits: $r: r_1 \to r_2$, $t: t_1 \to t_2$]

$$\int_{r_1}^{r_2} \frac{Q\,dr}{r^2} = \int_{t_1}^{t_2} (-K_0)(4\pi)(1+\alpha t+\beta t^2)\,dt$$

$$\Rightarrow Q\int_{r_1}^{r_2} \frac{dr}{r^2} = (-K_0)(4\pi)\int_{t_1}^{t_2} (1+\alpha t+\beta t^2)\,dt$$

$$\Rightarrow Q\left(-\frac{1}{r}\right) + C = (-4\pi K_0)\left(t+\frac{\alpha t^2}{2}+\frac{\beta t^3}{3}\right) \quad \text{--①}$$

Applying boundary condition at $r = r_1$, $t = t_1$

$$\Rightarrow C = (4\pi K_0)\left(t_1+\frac{\alpha t_1^2}{2}+\frac{\beta t_1^3}{3}\right) - \frac{Q}{r_1}$$

$$\Rightarrow \quad Q\left[\frac{-1}{r_2}-\left(\frac{-1}{r_1}\right)\right] = (-K_0)(4\pi)\left[(t_2-t_1)+\frac{\alpha}{2}(t_2-t_1)^2 + \frac{\beta}{3}(t_2-t_1)^3\right]$$

$$\Rightarrow \quad Q\left[\frac{1}{r_1}-\frac{1}{r_2}\right] = 4\pi K_0\left[(t_1-t_2) - \frac{\alpha}{2}(t_1-t_2)^2 + \frac{\beta}{3}(t_1-t_2)^3\right]$$

$$\Rightarrow \quad Q = \frac{4\pi K_0 r_1 r_2}{r_2-r_1}\left[(t_1-t_2)-\frac{\alpha}{2}(t_1-t_2)^2+\frac{\beta}{3}(t_1-t_2)^3\right]$$

$$\Rightarrow Q = \frac{4\pi K_0 r_1 r_2}{r_2-r_1}(t_1-t_2)\left[1-\frac{\alpha(t_1-t_2)}{2}+\frac{\beta}{3}(t_1-t_2)^2\right]$$

(5) SOLUTION:

For this case, the appropriate form of one dimensional steady state conduction equation (heat) is

$$\frac{d^2t}{dx^2}+\frac{q_g}{k}=0 \quad -- (1)$$

where q_g = Heat generation rate per unit volume.

$= q_0 e^{-rx}$ W/m³

$$\Rightarrow \quad \frac{d^2t}{dx^2} = -\frac{q_0 e^{-rx}}{k}$$

Integrating both sides:

$$\int \frac{d^2t}{dx^2} = \int \left(\frac{-q_0}{k}\right) e^{-rx}$$

$$\int \frac{d^2t}{dx^2}(dx) = \frac{-q_0}{k}\int e^{-rx}\,dx$$

$$\Rightarrow \quad \frac{dt}{dx} = \frac{-q_0}{k}\left(\frac{1}{(-r)} e^{-rx}\right) + c$$

$$\frac{dt}{dx} = \frac{q_0}{kr} e^{-rx} + c \quad \text{---} (1)$$

~~Since at "$x=0$" ie at the inside surface, the flat the flat plate is insulated, temperature gradient at that point equals '0'.~~

~~ie $\left.\frac{dt}{dx}\right|_{x=0} = 0$~~

At "$x=0$", the surface of flat plate is completely insulated and hence heat flow is zero.

From Fourier's law, $Q = -KA\left(\frac{dt}{dx}\right)$

At $x=0$, $Q=0 \Rightarrow \left.\frac{dt}{dx}\right|_{x=0} = 0$

Substituting in eq: (1), we get

$$c = 0 - \frac{q_0}{kr} e^{-r(0)} = \frac{-q_0}{kr}$$

So, eq-① becomes

$$\frac{dt}{dx} = \frac{g_o e^{-rx}}{Kr} - \frac{g_o}{Kr}.$$

$$= \frac{g_o}{Kr}[e^{-rx}-1]$$

Again integrating on both sides:

$$\int dt = \frac{g_o}{Kr}\int(e^{-rx}-1)\,dx$$

$$\Rightarrow t = -\frac{g_o}{Kr^2}[e^{-rx}+rx] + c.$$

we have, at $x=L$, $t=T_2$

$$\Rightarrow T_2 = -\frac{g_o}{Kr^2}[e^{-rL}+rL] + c$$

$$\Rightarrow c = T_2 + \frac{g_o}{Kr^2}[e^{-rL}+rL]$$

So Expression for temperature distribution is

$$t = -\frac{g_o}{Kr^2}[e^{-rx}+rx] + \frac{g_o}{Kr^2}[e^{-rL}+rL] + T_2.$$

(ii) At insulated surface, $x=0$.

$$\Rightarrow t_{ins} = \frac{-g_o}{Kr^2}[e^{-r(0)}+r(0)] + \frac{g_o}{Kr^2}[e^{-rL}+rL] + T_2$$

$$\Rightarrow t_{ins} = \frac{q_0}{k\gamma^2}\left[e^{-\gamma L} + \gamma L - 1\right] + T_2 .$$

⑥ SOLUTION :

For steady state one dimensional heat conduction in the radial direction, we have

$$\frac{d^2t}{dr^2} + \frac{1}{r}\frac{dt}{dr} + \frac{q_g}{k} = 0$$

$$\Rightarrow \frac{r\,d^2t}{dr^2} + \frac{dt}{dr} + \frac{q_g(r)}{k} = 0$$

$$\Rightarrow \frac{d}{dr}\left(r\frac{dt}{dr}\right) + \frac{q_g r}{k} = 0$$

Given $q_g = q_0\left[1 - \left(\frac{r}{R}\right)^2\right]$

$$\Rightarrow \frac{d}{dr}\left(r\frac{dt}{dr}\right) + \frac{q_0}{k}\left[1 - \left(\frac{r}{R}\right)^2\right] r = 0$$

Integrating :

$$r\frac{dt}{dr} + \frac{q_0}{k}\left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right] = C_1$$

$$\Rightarrow \frac{dt}{dr} + \frac{q_0}{k}\left[\frac{r}{2} - \frac{r^3}{4R^2}\right] = \frac{C_1}{r} \quad \text{--- ①}$$

From boundary conditions,

$$\frac{dt}{dr} = 0 \quad \text{at} \quad r = 0$$

so, from eq. 1 :

$$c_1 = 0$$

$$\Rightarrow \quad \frac{dt}{dr} + \frac{q_0}{K}\left[\frac{r}{2} - \frac{r^3}{4R^2}\right] = 0$$

Again Integrating,

$$\int dt + \frac{q_0}{K}\int\left[\frac{r}{2} - \frac{r^3}{4R^2}\right]dr = c$$

$$\Rightarrow \quad t + \frac{q_0}{K}\left[\frac{r^2}{4} - \frac{r^4}{16R^2}\right] = c_2$$

We have, at $r = 0$, $t = t_{max}$.

$$\Rightarrow \quad c_2 = t_{max}.$$

we get

$$t_{max} = t + \frac{q_0}{K}\left[\frac{r^2}{4} + \frac{r^4}{16R^2}\right]$$

where 't' is temperature at radius 'r'.